

Relativistic Brownian Motion in 3+1 Dimensions

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Abstract

We solve the problem of formulating Brownian motion in a relativistically covariant framework in $3 + 1$ dimensions. We obtain covariant Fokker-Planck equations with (for the isotropic case) a differential operator of invariant d'Alembert form. Treating the spacelike and timelike fluctuations separately in order to maintain the covariance property, we show that it is essential to take into account the analytic continuation of “unphysical” fluctuations. We discuss the notion of locality in this framework and possible implications for entangled states.

1 Introduction

The ideas of stochastic processes originated in the second half of the 19th century in thermodynamics, through the manifestation of the kinetic theory of gases. In 1905 A. Einstein [1] in his paper on Brownian motion provided a decisive breakthrough in the understanding of the phenomena. Moreover, it was a proof convincing physicists of the reality of atoms and molecules, the motivation for Einstein's work. It is interesting to note that Einstein predicted the so called Brownian motion of suspended microscopic particles not knowing that R. Brown first discovered it in 1827 [2]. The resemblance of the Schrödinger equation to the diffusion equation had lead physicists (including Einstein and Schrödinger) to attempt to connect quantum mechanics with an underlying stochastic process, the Brownian process.

Nelson [3], in 1966, constructed the Schrödinger equation from an analysis of Brownian motion by identifying the forward and backward average velocities of a Brownian particle with the real and imaginary parts of a wave function. He pointed out that the basic process involved is defined non-relativistically, and

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can be used if relativistic effects can be “safely” neglected. The development of a relativistically covariant formulation of Brownian motion could therefore provide some insight into the structure of a relativistic quantum theory.

Nelson pointed out that the formulation of his stochastic mechanics in the context of general relativity is an important open question [4]. The Riemannian metric spaces one can achieve, in principle, which arise due to nontrivial correlations between fluctuations in spatial directions, could, in the framework of a covariant theory of Brownian motion, lead to spacetime pseudo-Riemannian metrics in the structure of diffusion and Schrödinger equations. Morato and Viola [5] have recently constructed a relativistic quantum equation for a free scalar field. They assumed the existence of a 3D (spatial) diffusion in a comoving frame, a non-inertial frame in which the average velocity field of the Brownian particle (current velocity) is zero. In this frame the location of the Brownian particle in space experiences Brownian fluctuations parametrized by the proper time of the comoving observer. They interpreted possible negative 0-component current velocities with what they called ‘rare events’, which are time reversed Brownian processes (a peculiarity arising in the relativistic treatment). The equation they achieved this way is approximately the Klein-Gordon equation. It is important to note that in the inertial frame they do not obtain a normal diffusion. This is due to the fact that their process is stochastic only in three degrees of freedom and therefore is not covariant. In this paper we shall study a manifestly covariant form of Brownian motion.

Nelson himself [4] argued that Markov processes may lead to inconsistencies (which he demonstrated in entangled systems such as E.P.R [6]) unless a non-local theory is allowed, therefore according to his own preference for a local theory he suggested to consider non-Markovian theories. We shall treat this problem in our concluding section.

In a previous work [7] we introduced a new approach to the formulation of relativistic Brownian motion in 1+1 dimensions. The process we formulate is a straightforward generalization of the standard one dimensional diffusion to 1+1 dimensions (where the actual random process is thought of as a ‘diffusion’ in the time direction as well as in space), in an inertial frame. The equation achieved is an exact Klein-Gordon equation. It is a relativistic generalization of Nelson’s Brownian process, the Newtonian diffusion. In this paper we review the relativistic Brownian process in 1+1 dimensions [7] where the inclusion of both spacelike and timelike motion for the Brownian particle (event) is considered; if the timelike motion is considered as “physical” the “unphysical” spacelike motion is represented (through analytic continuation) by imaginary quantities. We extend the treatment to 3+1 dimensions using appropriate weights for the imaginary representations. The extension of the process to a *general* covariant form will be carried out in a succeeding paper. The complete formalism then can be used to construct relativistic general covariant diffusion and Schrödinger equations with pseudo-Riemannian metrics which follow from the existence of nontrivial correlations between the coordinate random variables.

Finally, we discuss the possible implications of the process we consider (i.e. a relativistic stochastic process with Markov property which preserves macro-

scopic Lorentz covariance) on the entangled system, where we claim that though fluctuations which exceed the velocity of light occur the macroscopic behavior dictated by the resulting Fokker-Planck equation is local.

2 The Problem of Assigning a Dynamical Evolution Parameter for the Relativistic Brownian Process and the Stueckelberg Formalism

Brownian motion, thought of as a series of “jumps” of a particle along its path, necessarily involves an ordered sequence. In the nonrelativistic theory, this ordering is naturally provided by the Newtonian time parameter. In a relativistic framework, the Einstein time t does not provide a suitable parameter. If we contemplate jumps in spacetime, to accommodate a covariant formulation, a possible spacelike interval between two jumps may appear in two orderings in different Lorentz frames. The introduction of proper time as a parameter for the RBP (Relativistic Brownian Process) is not adequate since in this case the second order correlations in the simplest case (i.e. for an isotropic homogeneous process with a diffusion constant σ^2) have the form:

$$E(\Delta x^\mu \Delta x^\mu) = 2\sigma^2 \Delta s \quad (1)$$

for each μ ; however, summing over μ ,

$$E(\Delta x_\mu \Delta x^\mu) \equiv \Delta s^2 \propto \Delta s, \quad (2)$$

where the first equality is by the definition of proper time and the second equality is due to the Brownian property expressed in Eq. (1); there is an obvious contradiction. We therefore adopt the invariant parameter τ as the dynamical variable for the Brownian process, first suggested in 1941 by E.C.G Stueckelberg [8]. The introduction of the notion of an invariant time τ permitted the discussion of world lines not monotonic in the ordinary (Einstein) time, t . On such world lines two points may have the same t coordinate and represent the occurrence of two particles existing at the same time. The time coordinate, t , may increase or decrease as τ evolves (just as a particle in Newtonian mechanics may move in the positive or negative spatial directions with time) representing particles and antiparticles respectively. In this way Stueckelberg was able to describe pair-creation and annihilation on a classical level and write a relativistic Schrödinger equation (with four coordinates of spacetime and the invariant time as a parameter).

In 1973 Horwitz and Piron [9] developed this concept, suggesting that all physical systems evolve through one universal invariant time, not affected by interaction or dynamical coordinate transformations, just as in the Newtonian theory. The theory formally resembles Newton’s theory with the most significant difference that Euclidian space is replaced by Minkowski spacetime.

The point particle, in Newtonian mechanics, represented by its spatial coordinates at each given time, is replaced by a new concept, the *event*; it is an object, evolving in spacetime, represented by four coordinates at each given (universal) time τ . The collection of the event's (four) positions in spacetime at all (universal) times forms the worldline. The time-coordinate of an event in a given inertial frame is the time in which, according to the clocks of this frame the event is detected, though it could have occurred, in principle, at a different (universal) time, τ . In particular let us consider a parabolic worldline (figure 1), describing pair annihilation, at a given time t_{OB} , such that the line $t = t_1$ intersects the worldline twice. At the time t_1 two events will be detected on the frame's clocks, one corresponding to a particle and the other to an antiparticle, each occurring at different (universal) time τ . The interpretation of the event going backwards in time as the antiparticle was given first by Stueckelberg and was later used by Feynman; it is now an accepted concept. However, in the parabolic worldline there are segments in which the event goes faster than light (either forward or backward in t). Do these segments have any physical representation? Should they be included in a physical theory? As we shall see in this paper the answer for the second question is positive in the formulation of our relativistic Brownian process, and the occurrence of this state of motion is dictated by the demand of achieving Lorentz invariance (more explicitly the d'Alembert) operator in the relativistic diffusion equation.

figure 1 : Particles and Events. A particle/antiparticle corresponds to a world line segment formed of a trajectory of an event whose time coordinate is monotonically increasing/decreasing with τ . At t_1 a particle and an anti-particle may be detected, both generated by an event B . The particle/antiparticle corresponds to the (left/right) branch of the parabola (within the forward/backward light cones) where the time coordinate is increasing/decreasing with τ . The segment of motion outside of the lightcones are tachyonic (spacelike) and are required by continuity of the worldline. These two points occur at different τ but at the same t . At times $t > t_{OB}$ (the turning point), the event B does not exist therefore the worldline generated by event B corresponds to a pair annihilation occurring at $t = t_{OB}$. An event, say, C may generate a worldline corresponding to a pair creation occurring at $t = t_{OC}$ and another event A may generate a monotonic worldline therefore corresponding to a single particle.

3 The Negative Correlation Problem

A second fundamental difficulty in formulating a covariant theory of Brownian motion lies in the form of the correlation function of the random variables of spacetime. The correlation function for the isotropic (non-relativistic) Wiener [10] process, is given by,

$$< \Delta x^i \Delta x^j > = \sigma^{ij} dt$$

$$\begin{aligned}\sigma^{ij} &= 2\sigma^2\delta^{ij} \\ i, j &= 1, 2, 3\end{aligned}\tag{3}$$

where the δ^{ij} generates the Euclidian structure for the manifold on which the Brownian evolution take place [11]. The straightforward covariant generalization to the relativistic case is,

$$< dw_\mu(\tau)dw_\nu(\tau') > = \begin{matrix} 0 & \tau \neq \tau' \\ 2\alpha^2\eta_{\mu\nu}d\tau & \tau = \tau' \end{matrix}\tag{4}$$

where $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, and therefore $< dw_0(\tau)dw_0(\tau) > < 0$, which is impossible. Let us consider, however, a process which is physically restricted to only to spacelike or timelike jumps. One may argue that Brownian motion in spacetime should be a generalization of the non-relativistic problem, constructed by observing the non-relativistic process from a moving frame according to the transformation laws of special relativity. Hence the process taking place in space in the non-relativistic theory would be replaced by a spacetime process in which the Brownian jumps are spacelike. The pure time (negative) self-correlation does therefore not occur. In order to meet this requirement, we shall use a coordinatization in terms of generalized polar coordinates which assures that all jumps are spacelike. Consider for example a relativistic Brownian probability density of the form $e^{-\frac{\mu^2}{ad\tau}}$, where μ is the invariant spacelike interval of the jump. This is a straightforward generalization of the standard Brownian process in 3D, which is generated by a probability density of the form $e^{-\frac{r^2}{ad\tau}}$, where r is the rotation invariant (i.e. the vector length) and a is proportional to the diffusion constant. We shall refer to this function as the relativistic Gaussian. As we shall see, a Brownian motion based on purely spacelike jumps does not, however, yield the correct form for an invariant diffusion process. We must therefore consider the possibility as well that, in the framework of relativistic dynamics, there are timelike jumps. The corresponding distribution would be expected to be of the form $e^{-\frac{\sigma^2}{bd\tau}}$, where σ is the invariant interval for the timelike jumps, and b is some constant. By suitably weighting the occurrence of the spacelike process (which we take for our main discussion to be “physical”, since its nonrelativistic limit coincides with the usual Brownian motion) and an analytic continuation of the timelike process, we show that one indeed obtains a Lorentz invariant Fokker-Planck equation in which the d’Alembert operator appears in place of the Laplace operator of the 3D Fokker-Planck equation. One may, alternatively, consider the timelike process as “physical” (as might emerge from a microscopic model with scattering) and analytically continue the spacelike (“unphysical”) process to achieve a d’Alembert operator with opposite sign.

4 Brownian motion in 1+1 dimensions

We consider a Brownian path in 1 + 1 dimensions generated by a stochastic differential (analogue to the Langevin equation [12] and Smoluchowsky process [13]), of the form:

$$dx^\mu(\tau) = \beta^\mu(x(\tau))d\tau + dw^\mu(\tau), \quad (5)$$

where dw is a random process, which is a relativistic generalization of the Wiener process, whose properties will be defined later, and β^μ is a deterministic field (the drift).

We start by considering the second order term in the series expansion of a function of position of the particle on the world line, $f(x^\mu(\tau) + \Delta x^\mu)$, involving the operator

$$\mathcal{O} = \Delta x^\mu \Delta x^\nu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}. \quad (6)$$

We have remarked that one of the difficulties in describing Brownian motion in spacetime is the possible occurrence of a negative value for the second moment of some component of the Lorentz four vector random variable. If the Brownian jump is timelike, or spacelike, however, the components of the four vector are not independent, but must satisfy the timelike or spacelike constraint. Such constraints can be realized by using parameterizations for the jumps in which they are restricted geometrically to be timelike or spacelike. We now separate the random jumps into space-like jumps and time-like jumps accordingly, i.e., for the spacelike jumps,

$$\Delta w^1 = \pm \mu \cosh \alpha, \quad \Delta w^0 = \mu \sinh \alpha \quad (7)$$

and for the timelike jumps,

$$\Delta w^1 = \sigma \sinh \alpha, \quad \Delta w^0 = \pm \sigma \cosh \alpha \quad (8)$$

Here we assume that the two sectors have the same distribution on the hyperbolic variable. We furthermore assume that μ, σ are generated by a relativistic Gaussian distribution, working in a Lorentz frame where the α distribution is assumed to be independent of μ, σ and is uniformly distributed on the restricted interval $[-L, L]$ (see discussion below) where L is arbitrary large. Therefore, in this frame $\langle \Delta w^\mu \rangle$ is 0 (this is true in all frames; see discussion in Section 5) and we pick a normalization such that (for any component) $\langle \Delta w^\mu \rangle \propto \Delta \tau^{\frac{n}{2}}$ so to first order in $\Delta \tau$ the contribution to $\langle \mathcal{O} \rangle$ comes only from $\langle \Delta w^\mu \Delta w^\nu \rangle$.

For a particle experiencing space-like jumps only, the operator \mathcal{O} takes the following form:

$$\mathcal{O}_{spacelike} = \mu^2 [\cosh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \sinh^2 \alpha \frac{\partial^2}{\partial t^2}] \quad (9)$$

If the particle undergoes time-like jumps only the operator \mathcal{O} takes the form:

$$\mathcal{O}_{timelike} = \sigma^2 [\sinh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \cosh^2 \alpha \frac{\partial^2}{\partial t^2}] \quad (10)$$

Since μ, σ and α are random processes, the average value of the operator \mathcal{O} is the sum of the two averages of Eq. (9) and Eq. (10). A difference between these two averages, leading to the d'Alembertian operator can only be obtained by considering the analytic continuation of the timelike process to the spacelike domain, choosing $\mu^2 = -\sigma^2$.

This procedure is analogous to the effect, well-known in relativistic quantum scattering theory, of a physical process in the crossed (t) channel on the observed process in the direct (s) channel. For example, in the LSZ formulation of relativistic scattering in quantum field theory [e.g. [14]], a creation operator in the “in” state may be moved to the left in the vacuum expectation value expression for the S -matrix, and an annihilation operator for the “out” state may be moved to the right. The resulting amplitude, identical to the original one in value, represents a process that is unphysical; its total “energy” (the sum of four-momenta squared) now has the wrong sign. Assuming that the S -matrix is an analytic function, one may then analytically continue the energy-momentum variables to obtain the correct sign for the physical process in the new channel. Although we are dealing with an apparently classical process, as Nelson has shown, the Brownian motion problem gives rise to a Schrödinger equation, and therefore contains properties of the differential equations of the quantum theory. We thus see the remarkable fact that one must take into account the physical effect of the analytic continuation of processes occurring in a non-physical, in this case timelike, domain, on the total observed behavior of the system.

In the timelike case, the velocity of the particle $\Delta w^1 / \Delta w^0 \leq 1$. We shall here use the dynamical association of coordinate increments with energy and momentum

$$E = M \frac{\Delta w^0}{\Delta \tau} \quad p = M \frac{\Delta w^1}{\Delta \tau}, \quad (11)$$

so that

$$\sigma^2 = \left(\frac{\Delta \tau}{M} \right)^2 (E^2 - p^2), \quad (12)$$

where M is a parameter of dimension mass associated with the Brownian particle. It then follows that $E^2 - p^2 = \left(\frac{M}{\Delta \tau} \right)^2 \sigma^2 > 0$. For the spacelike case, where $p/E > 1$, we may consider the transformation to an imaginary representation $E \rightarrow iE'$ and $p \rightarrow ip'$, for E', p' real¹, but $E^2 - p^2 \rightarrow p'^2 - E'^2 > 0$. In this case, we take the analytic continuation such that the magnitude of σ^2 remains unchanged, but can be called $-\mu^2$, so that $E'^2 - p'^2 = \mu^2$ with μ imaginary. The spacelike contributions are therefore obtained in this mapping by $E, p \rightarrow iE, ip$ and $\sigma \rightarrow i\mu$, assuring the formation of the d'Alembert operator when the timelike and spacelike fluctuations are added with equal weight (this

¹This transformation is similar to the continuation $p \rightarrow ip'$ in nonrelativistic tunnelling, for which the analytic continuation appears as an instanton.

equality is consistent with the natural assumption, in this case, of an equal distribution between spacelike and timelike contributions). The preservation of the magnitude of the interval reflects the conservation of a mass-like property which remains, as an intrinsic property of the particle, for both spacelike and timelike jumps. As mentioned before, one recalls the role of analytic continuation in quantum field theory; for the well known Wick rotation [e.g. [15]], however, in that case, only the 0-component is analytically continued and no clear direct physical idea or quantity is associated with it. In the RBP the identification of the imaginary 4-momentum is dynamical in origin. It is due to the Lorentz structure of spacetime, which distinguishes the transitions $\frac{\Delta \mathbf{x}}{\Delta t} > 1$ from those with $\frac{\Delta \mathbf{x}}{\Delta t} < 1$. Though one may object to the association of Δx^μ with a dynamical momentum (since the instantaneous derivative $\frac{dx^\mu}{d\tau}$ is not defined for a Brownian process) the Brownian motion could be understood as an approximation to a microscopic process, just as it appears in Einstein's famous work in 1905 [1], where it is assumed that the Brownian motion is produced by collisions. The effective conservation of $E^2 - \mathbf{p}^2$ as a real quantity in both timelike and spacelike processes suggests that it is a physical property which preserves its meaning in both sectors.

With these assumptions, the cross-term in hyperbolic functions cancels in the sum, which now takes the form

$$\mathcal{O} = \mu^2 \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right] \quad (13)$$

Taking into account the drift term in Eq. (5), one then finds the relativistic Fokker-Planck equation

$$\frac{\partial \rho(x, \tau)}{\partial \tau} = \left\{ -\frac{\partial}{\partial x^\mu} \beta^\mu + \langle \mu^2 \rangle \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} \right\} \rho(x, \tau), \quad (14)$$

where $\partial/\partial x^\mu$ operates on both β^μ and ρ .

We see that the procedure we have followed, identifying $\sigma^2 = -\mu^2$ and assuming equal weight, permits us to construct the Lorentz invariant d'Alembertian operator, as required for obtaining a relativistically covariant diffusion equation.

To see this process in terms of a higher symmetry, let us define the invariant $\kappa^2 \equiv E_t^2 - p_t^2 \geq 0$ for the timelike case; our requirement is then that $E_s^2 - p_s^2 = -\kappa^2$ for the spacelike case. In the framework of a larger group that includes κ as part of a three vector (E, κ, p) , the relation for the timelike case can be considered in terms of the invariant of the subgroup $O(1, 2)$, i.e., $E^2 - \kappa^2 - p^2$. The change in sign for the spacelike case yields the invariant $E^2 + \kappa^2 - p^2$; we designate the corresponding symmetry (keeping the order of E and p) as $O(2, 1)$. These two groups may be thought of as subgroups of $O(2, 2)$, where there exists a transformation which changes the sign of the metric of the subgroups holding the quantity κ^2 constant. The kinematic constraints we have imposed correspond to setting these invariants to zero (the zero interval in the $2+1$ and $1+2$ spaces).

The constraint we have placed on the relation of the timelike and spacelike invariants derives from the properties of the distribution function and the requirement of obtaining the d'Alembert operator, i.e, Lorentz covariance of the diffusion equation. It appears that in order for the Brownian motion to result in a covariant diffusion equation, the distribution function has a higher symmetry reflecting the necessary constraints. The transformations $E \rightarrow iE'$ and $p \rightarrow ip'$ used above would then correspond to analytic continuations from one (subgroup) sector to another. We shall see a similar structure in the $3+1$ case, where the groups involved can be identified with the symmetries of the $U(1)$ gauge fields associated with the quantum Stueckelberg-Schrödinger equation.

5 The Invariance Properties of the Process

Before formulating the $3+1$ dimensional Brownian process, let us investigate the Lorentz invariance of the process and the correlation functions. The averaging operations are summations with weights (probability) assigned to each quantity in the sum. The sums in the continuum are, of course, expressed by integrals. If we wish to assign a relativistic Gaussian distribution function then the hyperbolic angle integration is infinite unless we introduce a cutoff. The question then arises whether our process is invariant or not.

We will show that we can use an arbitrary non-invariant (scalar) probability distribution (for example, a cutoff on the hyperbolic angle) and still obtain Lorentz invariant averages, using the imaginary representations of the 'unphysical jumps'. For example, $\langle \Delta w^\mu \rangle$ stands for a summation with a scalar weight (given by the density) over all the vectors Δw^μ , in the domain. It is therefore a vector. Moreover under the imaginary representation of spacelike increments relative to the timelike ones (here we assume the timelike jumps physical), Δw^μ is a simple vector function over all spacetime which has the following form:

$$\begin{aligned}\Delta w^\mu &= \Delta w'^\mu, \Delta w'^\mu \text{ timelike} \\ \Delta w^\mu &= i\Delta w'^\mu, \Delta w'^\mu \text{ spacelike.}\end{aligned}\tag{15}$$

where the $\Delta w'^\mu$ are real. The quantity $\langle \Delta w^\mu \rangle$ (formally written as a discrete sum) is given therefore by:

$$\begin{aligned}\langle \Delta w^\mu \rangle &= \sum P(\Delta w) \Delta w^\mu = \sum_{\text{timelike}} P'(\Delta w') \Delta w'^\mu + i \sum_{\text{spacelike}} P'(\Delta w') \Delta w'^\mu = \\ &= \langle \Delta w'^\mu \rangle_{\text{timelike}} + i \langle \Delta w'^\mu \rangle_{\text{spacelike}}\end{aligned}\tag{16}$$

where $P(\Delta w)$ (or $P'(\Delta w')$) is the probability(weight) of having the vector Δw (or $\Delta w'$). The two vectors in the last equality in Eq. (16) are just normal

Lorentz vectors. If we now pick a distribution in a given frame for which the average of each of them (independent of the other) is zero then $\langle \Delta w^\mu \rangle = 0$ is true in all frames since the 0-vector is Lorentz invariant.

Building the second correlation, with the assumption of no correlation between spacelike jumps and timelike jumps, we find:

$$\begin{aligned}
\langle \Delta w^\mu \Delta w^\nu \rangle &= \sum P(\Delta w) \Delta w^\mu \Delta w^\nu = \sum_{\text{timelike}} P'(\Delta w') \Delta w'^\mu \Delta w'^\nu + \\
&+ i^2 \sum_{\text{spacelike}} P'(\Delta w') \Delta w'^\mu \Delta w'^\nu \\
&= \sigma_{\text{timelike}}^{\mu\nu} - \sigma_{\text{spacelike}}^{\mu\nu} \propto \eta^{\mu\nu} D\Delta\tau
\end{aligned} \tag{17}$$

where $\sigma^{\mu\nu}$ is the correlation tensor in each case. From the definition of $\Delta w'^\mu$ (a four vector) it follows that $\sigma^{\mu\nu}$ are real Lorentz tensors. The last equality in Eq. (17) is a demand that could be achieved for the general 1+n case, by assuming that in a given frame there is an invariant Gaussian distribution where the distribution is uniform in all angles and that there is a cutoff in the hyperbolic angle. The sum of the two covariant tensors (each a result of summation on different sectors) is a Lorentz invariant tensor. The higher correlation functions do not interest us since they are of higher order in ρ and therefore in $\Delta\tau$ and do not contribute to the Fokker-Plank equation.

6 The Notion of ‘Jumps’ Versus a Continuous Process

The mapping given in Eq. (15) leads necessarily to a deviation from the standard mathematical formulation of Brownian motion. There the probability that a particle starting at x at time τ ending at x' at time τ' is equal to the probability that the particle starts at x at time τ passing through any possible intermediate point x'' at time $\tau'' < \tau'$ and going from there to the point x' at time τ' . This property is expressed in the Chapman-Kolmogorov equation [e.g. [16]],

$$p(x, \tau, x', \Delta\tau') = \int_{\mathcal{R}} p(x, \tau, x'', \tau'') p(x'', \tau'', x', \tau') d^4 x'', \tag{18}$$

In the relativistic formulation the vector $\Delta w' = x - x'$ could be a timelike vector therefore resulting in a real valued vector Δw according to the mapping in Eq. (15) However, the two intermediate vectors $\Delta w'_1 = x - x''$ and $\Delta w'_2 = x'' - x'$ could be spacelike, and take the event out of the real manifold into a complex valued coordinate. In this case the Chapman-Kolmogorov equation does not hold, and the event may be found outside of the real manifold. In order to build a consistent process one must adopt the concept of ‘Brownian jumps’ which could be a result for example of a process in which the event (similar to Einstein’s

original construction) undergoes collisions and for each collision, or ‘jump’ the mapping in Eq. (15) holds. Therefore at each point in the physical manifold the event may take any increment spacelike or timelike (with a possible complex valued contribution to the averages). However, although the vector leading from the initial point, say O , to the end point, A , may be spacelike and therefore be represented as an imaginary vector it is understood that the event arrives at the real spacetime point A , never leaving the real spacetime. This structure separates the two manifolds, spacetime which is real and represents the physical coordinates of the event and a complex space representing the processes the event undergoes going from one point to another. This structure differs in that sense from the mathematical formulation due to Wiener and others, but still it can be shown that the process is invariant on the average under decomposition into shorter time subprocesses. In other words, we consider the event starting at some arbitrary point, and going for some time $\Delta\tau$. We next decompose the time interval into M intervals, so that:

$$\sum_{i=1}^M \Delta\tau_i = \Delta\tau$$

We consider then the expression appearing in the Fokker-Plank equation $\langle \Delta x^\mu \Delta x^\nu \rangle$:

$$\langle \Delta x^\mu \Delta x^\nu \rangle = \left\langle \left(\sum_{i=1}^M \Delta x_i^\mu \right) \left(\sum_{j=1}^M \Delta x_j^\nu \right) \right\rangle \quad (19)$$

Since we assume that any two non-equal time jumps are not correlated, i.e. $\langle \Delta x_i \Delta x_j \rangle = 0$ for $i \neq j$, which leaves only the equal time averages in the sum,

$$\langle \Delta x^\mu \Delta x^\nu \rangle = \sum_{i=1}^M \langle \Delta x_i^\mu \Delta x_i^\nu \rangle = \sigma^2 \eta^{\mu\nu} \sum_{i=1}^M \Delta\tau_i = \sigma^2 \eta^{\mu\nu} \Delta\tau \quad (20)$$

where σ^2 is the diffusion constant and we used Eq. (17) going from the second piece in the equality to the third

The notion of ‘jumps’ stimulates the consideration of discrete processes, which can also be formulated within the relativistic framework and leads, under certain assumptions, to a covariant Fokker-Plank equation. For example let us assume a physical process in which the ‘jumps’ occur in a very ordered way every τ_J seconds with a very small time spread (i.e. a very small probability that a collision occurs within a time different significantly from τ_J). Then, averaging the ‘jumps’ over a period $\tau \gg \tau_J$ leads to:

$$\begin{aligned} \langle \Delta w^\mu \Delta w^\nu \rangle &\cong N \sigma^2 \eta^{\mu\nu} \tau_J \\ N \tau_J &< \tau < (N+1) \tau_J \end{aligned} \quad (21)$$

This result is due to the fact that under our assumptions during the time τ , N single ‘jumps’ within separation of each other of τ_J occurred. The average in

Eq. (21) does not change when τ changes in less than τ_J ; however if τ_J is small then one can replace Eq. (21), with,

$$\langle \Delta w^\mu \Delta w^\nu \rangle \cong \sigma^2 \eta^{\mu\nu} \tau \quad (22)$$

Therefore we recover the standard result for Brownian motion. However there is one very important difference which is the fact that τ can be taken to be finitely small where in the standard Brownian process τ can be actually taken to zero. This implies that higher order derivative terms enter into the resulting 'diffusion' equation. For example for an isotropic homogeneous Gaussian distribution there will be additional even order derivative operators beyond the second order (d'Alembert) with coefficients $\sigma^n \tau_J (2n - 1)$ where n is the (even) order of the differential operator. Since both τ_J and σ^2 are small these operators could be neglected in general, though there might be special configurations in which their effect may be significant. In the following we assume that the τ_J is very small compared with the macroscopic scale and that the 'jumps' are practically ordered with zero spread, thus the approximation in Eq. (22) is valid and no higher order terms are considered.

7 Brownian motion in 3 + 1 dimensions

In the 3 + 1 case, we again separate the jumps into timelike and spacelike types. The spacelike jumps may be parameterized, in a given frame, by

$$\begin{aligned} \Delta w^0 &= \mu \sinh \alpha \\ \Delta w^1 &= \mu \cosh \alpha \cos \phi \sin \vartheta \\ \Delta w^2 &= \mu \cosh \alpha \sin \phi \sin \vartheta \\ \Delta w^3 &= \mu \cosh \alpha \cos \vartheta \end{aligned} \quad (23)$$

We assume the four variables $\mu, \alpha, \vartheta, \phi$ are independent random variables. In addition we demand in this frame that ϑ and ϕ are uniformly distributed in their ranges $(0, \pi)$ and $(0, 2\pi)$, respectively. In this case, we may average over the trigonometric angles, i.e., ϑ and ϕ and find that:

$$\begin{aligned} \langle \Delta w^{1^2} \rangle_{\phi, \vartheta} &= \langle \Delta w^{2^2} \rangle_{\phi, \vartheta} = \langle \Delta w^{3^2} \rangle_{\phi, \vartheta} = \frac{\mu^2}{3} \cosh^2 \alpha \\ \langle \Delta w^{0^2} \rangle_{\phi, \vartheta} &= \mu^2 \sinh^2 \alpha \end{aligned} \quad (24)$$

We may obtain the averages over the trigonometric angles of the timelike jumps by replacing everywhere in Eq. (24)

$$\begin{aligned} \cosh^2 \alpha &\leftrightarrow \sinh^2 \alpha \\ \mu^2 &\rightarrow \sigma^2 \end{aligned}$$

to obtain

$$\begin{aligned} \langle \Delta w^{1^2} \rangle_{\phi, \vartheta} &= \langle \Delta w^{2^2} \rangle_{\phi, \vartheta} = \langle \Delta w^{3^2} \rangle_{\phi, \vartheta} = \frac{\sigma^2}{3} \sinh^2 \alpha \\ \langle \Delta w^{0^2} \rangle_{\phi, \vartheta} &= \sigma^2 \cosh^2 \alpha, \end{aligned} \quad (25)$$

where σ is a real random variable, the invariant timelike interval. Assuming, as in the 1 + 1 case, that the likelihood of the jumps being in either the spacelike or (virtual) timelike phases are equal, and making an analytic continuation for which $\sigma^2 \rightarrow -\lambda^2$, the total average of the operator \mathcal{O} , including the contributions of the remaining degrees of freedom μ, λ and α is

$$\begin{aligned} \langle \mathcal{O} \rangle &= (\langle \mu^2 \rangle \langle \sinh^2 \alpha \rangle - \langle \lambda^2 \rangle \langle \cosh^2 \alpha \rangle) \frac{\partial^2}{\partial t^2} + \\ &\quad \frac{1}{3} (\langle \mu^2 \rangle \langle \cosh^2 \alpha \rangle - \langle \lambda^2 \rangle \langle \sinh^2 \alpha \rangle) \Delta \end{aligned} \quad (26)$$

If we now insist that the operator $\langle \mathcal{O} \rangle$ is invariant under Lorentz transformations (i.e. the d'Alembertian) we impose the condition

$$\begin{aligned} \langle \mu^2 \rangle \langle \sinh^2 \alpha \rangle - \langle \lambda^2 \rangle \langle \cosh^2 \alpha \rangle &= \\ -\frac{1}{3} (\langle \mu^2 \rangle \langle \cosh^2 \alpha \rangle - \langle \lambda^2 \rangle \langle \sinh^2 \alpha \rangle) \end{aligned} \quad (27)$$

Using the fact that $\langle \cosh^2 \alpha \rangle - \langle \sinh^2 \alpha \rangle = 1$, and defining $\gamma \equiv \langle \sinh^2 \alpha \rangle$, we find that

$$\langle \lambda^2 \rangle = \frac{1 + 4\gamma}{3 + 4\gamma} \langle \mu^2 \rangle \quad (28)$$

The Fokker-Planck equation then takes on the same form as in the 1 + 1 case, i.e., the form Eq. (14). We remark that for the 1 + 1 case, one finds in the corresponding expression that the 3 in the denominator is replaced by unity, and the coefficients 4 are replaced by 2; in this case the requirement reduces to $\langle \mu^2 \rangle = \langle \lambda^2 \rangle$ and there is no γ dependence.

We see that in the limit of a uniform distribution in α , for which $\gamma \rightarrow \infty$,

$$\langle \lambda^2 \rangle \rightarrow \langle \mu^2 \rangle. \quad (29)$$

In this case, the relativistic generalization of nonrelativistic Gaussian distribution of the form $e^{-\frac{r^2}{4t}}$ is $e^{-\frac{\mu^2}{4\tau}}$, which is Lorentz invariant.

The limiting case $\gamma \rightarrow 0$ corresponds to a stochastic process in which in the spacelike case there are no fluctuations in time, i.e., the process is that of a nonrelativistic Brownian motion. For the timelike case (recall that we have assumed the same distribution function over the hyperbolic variable) this limit implies that the fluctuations are entirely in the time direction. The limit $\gamma \rightarrow \infty$ is Lorentz invariant, but the limit $\gamma \rightarrow 0$ can clearly be true only in a particular frame.

8 The Markov Relation and the 4D Gaussian Process

In developing the previous ideas leading to the formulation of a RBP, we assumed that the probability distribution is consistent with the Markov property expressed in the Chapman-Kolmogorov equation [e.g. [16]] However, for the relativistic Gaussian it is not clear whether Eq. (18) holds. Therefore we now consider an alternative process, using the ideas developed above, resulting eventually in the Klein-Gordon equation. Let us consider a 2D Gaussian process generated by a distribution of the form :

$$p(w, d\tau) = \frac{1}{2\pi D d\tau} \exp\left(\frac{-\Delta w_0^2 - \Delta w_1^2}{2D d\tau}\right) \quad (30)$$

This distribution corresponds to a Markov process, a standard normalized Wiener process, where D is the diffusion constant. We now use the coordinate representation given in Eq. (7) and Eq. (8) for the timelike and spacelike sectors to transform the distribution function in Eq. (30) in both sectors to (use μ^2 in both cases):

$$\frac{1}{2\pi D d\tau} \exp\left(\frac{-\mu^2 \cosh 2\alpha}{2D d\tau}\right) \quad (31)$$

where timelike ‘jumps’ are physical and the measure for both sectors is $\mu d\mu d\alpha$. Then, following Section 5, using Eq. (15), we get for the combination of the timelike and spacelike contributions (with the appropriate sign) of the averages, say, Δw_0^2 ,

$$\begin{aligned} \langle \Delta w_0^2 \rangle &= \frac{1}{2\pi D d\tau} \int_0^\infty \int_{-\infty}^\infty \mu^3 \exp\left(\frac{-\mu^2 \cosh 2\alpha}{2D d\tau}\right) d\mu d\alpha = \\ &= \frac{1}{\pi} D d\tau \end{aligned} \quad (32)$$

where we integrated over μ first, using

$$\int_0^\infty \mu^n \exp(-a\mu^2) d\mu = \frac{\Gamma(\frac{n+1}{2})}{2a^{(n+1)/2}} \quad (33)$$

and then integrated over α using

$$I_2 \equiv \int_{-\infty}^\infty \frac{d\alpha}{\cosh^2 2\alpha} = 1 \quad (34)$$

In a similar way one finds that (using Eq. (15) leading to the negative sign)

$$\langle \Delta w_1^2 \rangle = -\frac{1}{\pi} D d\tau \quad (35)$$

Since the probability distribution Eq. (30) is symmetric in Δw_i in each sector $\langle \Delta w_0 \Delta w_1 \rangle = 0$ as well as the first moments. Therefore we get in this particular frame a d’Alembertian. However, following the results of Section 5 we see

that it is an invariant result in all Lorentz frames (though in other frames the distribution may not appear to be Gaussian).

Next we consider the application of the 4D form of Eq. (30)

$$p(w, d\tau) = \frac{1}{4\pi^2 D^2 (d\tau)^2} \exp\left(\frac{-\Delta w_0^2 - \Delta w_1^2 - \Delta w_2^2 - \Delta w_3^2}{2D d\tau}\right) \quad (36)$$

with measure $\mu^3 d\mu \cosh^2 \alpha \sin \theta d\theta d\alpha d\phi$ for the spacelike sector and $\mu^3 d\mu \sinh^2 \alpha \sin \theta d\theta d\alpha d\phi$ for the timelike sector.

However, now calculating $\langle \Delta w_0^2 \rangle$ for the timelike case, after averaging over the spatial angles θ and ϕ we find, using Eq. (23),

$$\langle \Delta w_0^2 \rangle = \frac{1}{\pi D^2 (d\tau)^2} \int_0^\infty \int_{-\infty}^\infty \mu^5 \exp\left(\frac{-\mu^2 \cosh 2\alpha}{2D d\tau}\right) \cosh^2 \alpha \sinh^2 \alpha d\mu d\alpha \quad (37)$$

and for the spacelike case we get the same result since the spacelike parametrization of Δw_0^2 is proportional to $\sinh^2 \alpha$ and the spacelike volume element is proportional to $\cosh^2 \alpha$. Therefore if we use Eq. (15), adding the contribution of the two sectors one obtains a complete cancellation to zero. In order to avoid this we extend Eq. (15) to the form

$$\begin{aligned} \Delta w^\mu &= \Delta w'^\mu, \Delta w'^\mu \text{ timelike} \\ \Delta w^\mu &= i\lambda \Delta w'^\mu, \Delta w'^\mu \text{ spacelike} \end{aligned} \quad (38)$$

Before completing the calculation, we discuss the inclusion of the factor λ , in Eq. (38). Let us consider a classical (i.e. non-stochastic) event with a given value $m^2 \equiv \Delta w_\mu \Delta w^\mu$, moving in a timelike direction. It then changes its state of motion and starts moving in a spacelike direction; according to Eq. (38) m^2 changes into $\lambda^2 m^2$. Moreover, though the event may move according to a Gaussian distribution which makes no distinction between timelike and spacelike motions, the outcome of this motion as represented by the Δw , in Eq. (15); Eq. (38) does distinguish the two phases of motion. We shall see that a specific value of λ is required for the realization of the Fokker-Planck equation.

The w manifold is complex and it is a function of the motion on the real manifold w' . Our macroscopic (physical) equations are written on the real plane of the w manifold. One can then visualize the flow of an event in spacetime similar to a motion of a particle in a cloud chamber. There as the particle moves the gas condenses, therefore the particle leaves a track. The track itself is not the particle but a result of the actual motion of the particle and its interaction with the gas in the cloud chamber. The track in the cloud chamber is analogous to the complex representation we use for the ‘jumps’.

We calculate first the expectation $\langle \Delta w_0^2 \rangle$ which is the total expectation, summed over the timelike and spacelike sectors. Averaging over the spherical angles θ, φ we get using Eq. (37) and Eq. (38),

$$\langle \Delta w_0^2 \rangle = \frac{1 - \lambda^2}{\pi} D^2 (d\tau)^2 \int_0^\infty \int_{-\infty}^\infty \mu^5 \exp\left(\frac{-\mu^2 \cosh 2\alpha}{2D d\tau}\right) \cosh^2 \alpha \sinh^2 \alpha d\mu d\alpha =$$

$$= \frac{8(1-\lambda^2)}{\pi} D d\tau \int_{-\infty}^{\infty} \frac{\cosh^2 \alpha \sinh^2 \alpha}{\cosh^3 2\alpha} d\alpha \quad (39)$$

where Eq. (33) was used in the μ integration leading to the last equality in Eq. (39). Using

$$\begin{aligned} \cosh^2 \alpha &= \frac{1}{2}(\cosh 2\alpha + 1) \\ \sinh^2 \alpha &= \frac{1}{2}(\cosh 2\alpha - 1) \end{aligned} \quad (40)$$

in Eq. (39) and integrating over α we get

$$\langle \Delta w_0^2 \rangle = \frac{(1-\lambda^2)}{\pi} D d\tau \frac{\pi}{2} \quad (41)$$

where we used

$$\begin{aligned} I_1 &\equiv \int_{-\infty}^{\infty} \frac{d\alpha}{\cosh 2\alpha} = \frac{\pi}{2} \\ I_3 &\equiv \int_{-\infty}^{\infty} \frac{d\alpha}{\cosh^3 2\alpha} = \frac{\pi}{4} \end{aligned} \quad (42)$$

We now calculate the expectation of $\langle \Delta w_1^2 \rangle$. Averaging over the spherical angles θ, φ we get, using Eq. (37) and Eq. (38),

$$\begin{aligned} \langle \Delta w_1^2 \rangle &= \frac{1}{3\pi} D^2 (d\tau)^2 \int_0^\infty \int_{-\infty}^\infty \mu^5 \exp\left(\frac{-\mu^2 \cosh 2\alpha}{2Dd\tau}\right) (\sinh^4 \alpha - \lambda^2 \cosh^4 \alpha) d\mu d\alpha = \\ &= \frac{8}{3\pi} D d\tau \int_{-\infty}^\infty \frac{(\sinh^4 \alpha - \lambda^2 \cosh^4 \alpha)}{\cosh^3 2\alpha} d\alpha \end{aligned} \quad (43)$$

Using Eq. (40), Eq. (34), Eq. (42), We get after integration over α ,

$$\langle \Delta w_1^2 \rangle = \frac{D d\tau}{\pi} \frac{1}{3} \left[(1-\lambda^2) \frac{3\pi}{2} - 4(1+\lambda^2) \right] \quad (44)$$

In order to obtain the d'Alembertian we insist that $\langle \Delta w_1^2 \rangle = -\langle \Delta w_0^2 \rangle$, which leads to

$$\lambda^2 = \frac{3\pi - 4}{3\pi + 4} \quad (45)$$

Finally, substituting (for example) Eq. (45) in Eq. (41) We find

$$\langle \Delta w^\mu \Delta w^\nu \rangle = \eta^{\mu\nu} \frac{4D}{3\pi + 4} d\tau = \eta^{\mu\nu} \check{D} d\tau \quad (46)$$

where \check{D} , is the actual effective diffusion constant defined by

$$\check{D} \equiv \frac{4D}{3\pi + 4} \quad (47)$$

9 Discussion and Conclusions

We have constructed a relativistic generalization of Brownian motion, using the invariant world-time, τ , to order the Brownian fluctuations, and separated consideration of spacelike and timelike jumps to avoid the problems of negative second moments which might otherwise follow from the Minkowski signature. Associating the Brownian fluctuations with an underlying dynamical process, one may think of γ discussed in the $3+1$ case as an order parameter, where the distribution function (over α), associated with the velocities, is determined by the temperature of the underlying dynamical system (the result for the $1+1$ case is independent of the distribution on the hyperbolic variable). More generally it is suggestive to consider the possible thermodynamical effects of the ‘medium’ generating the relativistic Brownian fluctuations, following similar steps taken by Einstein [1] in his famous work and verify whether any physical effect can be predicted.

At equilibrium, where $\partial\rho/\partial\tau = 0$, the resulting diffusion equation turns into a classical wave equation which, in the absence of a drift term K^μ , is the wave equation for a massless field. An exponentially decreasing distribution in τ of the form $\exp -\kappa\tau$ would correspond to a Klein-Gordon equation for a particle in a tachyonic state (mass squared $-\kappa$), for physical spacelike motion and for physical timelike motion to a particle with mass squared κ .

Choosing a cutoff in the hyperbolic angle, one finds a covariant moment and therefore, covariant differential operators. However the underlying process is not invariant, thus one can think of a special frame in which the hyperangular distribution is uniformly distributed around 0. Boosting breaks the symmetry of the hyperangular distribution, but since the averages are tensor quantities the invariance properties are conserved, and therefore the Fokker-Planck equation (leading to the quantum equation) is invariant. This property is also used to construct the 4D Gaussian process.

It was shown that a (Euclidian) Gaussian process with an appropriate (weighted) complex representation for the timelike and spacelike random motions can be used to achieve the covariant quantum equation, with the assurance that it is Markovian, (it is a relativistic generalization of the Wiener process). This leads to a ‘cloud chamber-like’ picture in which the event as it evolves leaves a track (carries a real or an imaginary phase), which is a representation of the actual motion, distinguishing the timelike and spacelike motion.

In the classical Stueckelberg theory the timelike (forward or backward) propagation is associated with the standard particle or antiparticle interpretation, where spacelike propagation is needed whenever one discuss classical pair creation or annihilation (with continuous passage from forward to backward motion in time). This suggests that the spacelike process may be associated with the annihilation and creation of pairs. Moreover, though the resulting macroscopic equation (i.e. on the level of the Fokker-Planck equation) is local and causal in the spacetime variables, the underlying microscopic process (i.e., on the level of the Brownian fluctuations) is not. It is however local and causal in τ even at the microscopic level. This non-locality in t ‘microscopically’ may lead to a

mechanism providing correlations for the entangled state system.

Nelson has shown that non-relativistic Brownian motion can be associated with a Schrödinger equation. Equipped with the procedures we presented here, constructing relativistic Brownian motion, Nelson's methods can be generalized. One then can construct relativistic equations of Schrödinger (Schrödinger-Stueckelberg) type. The eigenvalue equations for these relativistic forms are also Klein-Gordon type equations. Moreover one can also generalize the case where the fluctuations are not correlated in different directions into the case where correlations exist, as discussed by Nelson for three dimensional Riemannian spaces. In this case the resulting equation will be a quantum equation in a curved Riemannian spacetime; as we have pointed out, the eikonal approximation to the solutions of such an equation contains the geodesic motion of classical general relativity. The medium supporting the Brownian motion may be identified with an "ether" for which the problem of local Lorentz symmetry is solved. This study opens up several tracks of possible research. Nelson, discussing the E.P.R [4] system confronted the fact that such a system may be either described by a non-local Markov process or a local non-Markov process. The Markov process is simple to implement but Nelson was disturbed by the introduction of non-local interaction. However, the non-Markov process is very difficult to apply. The RPB developed here may bridge the two possibilities since an ordered (causal) Markov process in τ may appear to be a non-Markovian (or possibly non-local and certainly non causal) process in t . For example for the Gaussian process, looking for the probability of finding the event changing its spatial position Δx after Δt has passed, one may integrate Eq. (30) over all τ . This results however, in $\frac{1}{\Delta x^2 + \Delta t^2}$ which is not integrable and therefore can not be normalized. This however is not surprising, since the probability of finding the event in Δx after Δt is not well defined (there may be several values of Δx for a given Δt). For example the number of particles, as in Stueckelberg original construction [8] depends on the trajectory through which the point is reached. Defining an appropriate one particle probability resulting from the initial process occurring in τ demands a restriction of the sample space before integrating over τ i.e., using the conditional probability restricted to processes for which the event's t coordinate is monotonic in τ (no pairs are created).

Finally we would like to point out that generating a covariant quantum equation through an RBP leads to a possible relation between quantum mechanics and gravitation. In the context of this work, the metric of gravity can appear as an anisotropy in the correlations that lead to quantum equations for which the ray, or eikonal approximation, corresponds to the classical geodesic flow of general relativity. It furthermore appears interesting to generalize Einstein's famous work on this process introducing thermodynamic concepts to the resulting geometrical structure of the theory.

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